Residues of functions of Cayley-Dickson variables and Fermat's last theorem.

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Abstract

Function theory of Cayley-Dickson variables is applied to Fermat's last theorem. For this the homotopy theorem, Rouché's theorem and residues of meromorphic functions over Cayley-Dickson algebras are used. A special meromorphic function of Cayley-Dickson variables is constructed and its properties are investigated.

1 Introduction.

Analytical methods are frequently used in number theory, particularly of complex analysis [5, 6, 15]. It is logical to use hypercomplex over Cayley-Dickson algebras analysis to problems of number theory. One of such interesting objects is Fermat's last theorem [6, 15, 16]. Its existing proof is very long and complicated. This article is devoted to a rather short and clear demonstration of Fermat's last theorem with the help of (super-)analysis over Cayley-Dickson algebras.

Such algebras have a long history, because quaternions were first introduced by W.R. Hamilton in 1843. He had planned to use them for problems of mechanics and mathematics [4]. Their generalization known as the octonion algebra was introduced by J.T. Graves and A. Cayley in 1843-45. Then Dickson had investigated more general algebras known now as the Cayley-Dickson algebras [1, 2, 7].

This work continues previous articles of the author. In those articles (super)-differentiable functions of Cayley-Dickson variables and their non-commutative line integrals were investigated [10, 11, 12, 13]. Meromorphic functions of Cayley-Dickson variables, their arguments and residues were studied in the papers [11, 10, 14]. Super-differentiability or z-differentiability (or (z, z^*)- or z^*-differentiability) over Cayley-Dickson algebras $\mathcal{A}_r$ is a specific differentiation of the algebra of locally converging formal power series on an open or canonical closed domain U in $\mathcal{A}_r$ with prescribed order of multiplication of the Cayley-Dickson variable $z \in U$ (or $(z, z^*) \in U$ or $z^* \in U$ respectively) and constants from $\mathcal{A}_r$ which are ordered in each addendum, where $z^* = \tilde{z}$ denotes the conjugated Cayley-Dickson number z. Each locally analytic function is considered on an open neighborhood of a point of a domain as a given phrase by z (or (z, z^*) or z^* correspondingly) that to preserve algebraic features, because functions in the set-theoretic sense do not bear an algebraic information.

The Cayley-Dickson algebras $\mathcal{A}_r$ have the even generator $i_0 = 1$ and the purely imaginary odd generators $i_1, ..., i_{2^r-1}$, $2 \le r$, $i_k^2 = -1$ and $i_0 i_k = i_k$ and $i_k i_l = -i_l i_k$ for each $1 \le k \ne l$. For $3 \le r$ the multiplication of these generators is generally non-associative, so they form not a group, but a non-commutative quasi-group with the property of alternativity $i_k(i_k i_l) = (i_k^2) i_l$ and $(i_l i_k) i_k = i_l (i_k^2)$ instead of associativity.

The purpose of this paper is in application of developed earlier technique of residues of meromorphic functions of Cayley-Dickson variables to Fermat's last theorem. For this the argument principle, homotopy theorem, Rouché's theorem and residues of meromorphic functions over Cayley-Dickson algebras are used. A special meromorphic function of Cayley-Dickson variables is constructed and its properties are investigated. It is demonstrated that this approach is effective.

Main results of this paper are obtained for the first time. In this article notations and definitions of previous papers cited above are used.

2 Special meromorphic function and its poles.

To avoid misunderstanding we first recall some notations and basic facts in §§1-6 from [10, 11, 14].

1. Cayley-Dickson algebras $\mathcal{A}_r$ form the sequence so that $\mathcal{A}_{r+1}$ is obtained from the preceding $\mathcal{A}_r$ with the help of the so called doubling procedure [1, 7, 8]. Therefore, the natural embeddings $\mathcal{A}_r \hookrightarrow \mathcal{A}_{r+1} \hookrightarrow ...$ are induced. It is convenient to put: $\mathcal{A}_0 = \mathbf{R}$ for the real field, $\mathcal{A}_1 = \mathbf{C}$ for the complex field, $\mathcal{A}_2 = \mathbf{H}$ denotes the quaternion skew field, $\mathcal{A}_3 = \mathbf{O}$ is the octonion algebra, $\mathcal{A}_4$ denotes the sedenion algebra. The quaternion skew field $\mathbf{H}$ is associative, but non-commutative. The octonion algebra $\mathbf{O}$ is the alternative division algebra with the multiplicative norm. The sedenion algebra and Cayley-Dickson algebras of higher order $r \ge 4$ are not division algebras and have not any non-trivial multiplicative norm. Nevertheless, they are power-associative, that is $z^n z^m = z^{n+m}$ for any natural numbers n and m and each Cayley-Dickson number z, where $z^n = (z(...(zz)...))$ is the n-th power of z (see also [1, 2, 7]). The norm in the Cayley-Dickson algebra $\mathcal{A}_r$ is defined by the equality $|z|^2 = zz^*$.

We recall the doubling procedure for the Cayley-Dickson algebra $\mathcal{A}_{r+1}$ from $\mathcal{A}_r$, because it is frequently used. Each Cayley-Dickson number $z \in \mathcal{A}_{r+1}$ is written in the form $z = \xi + \eta \mathbf{l}$, where $\mathbf{l}^2 = -1$, $\mathbf{l} \notin \mathcal{A}_r$, $\xi, \eta \in \mathcal{A}_r$. The addition of such numbers is componentwise. The conjugate of any Cayley-Dickson number z is prescribed by the formula:

(1) $z^* := \tilde{z} := \xi^* - \eta \mathbf{l}$.

The multiplication in $\mathcal{A}_{r+1}$ is defined by the following equation:

(2) $(\xi + \eta \mathbf{l})(\gamma + \delta \mathbf{l}) = (\xi\gamma - \tilde{\delta}\eta) + (\delta\xi + \eta\tilde{\gamma})\mathbf{l}$

for each ξ, η, γ, $\delta \in \mathcal{A}_r$, $z := \xi + \eta \mathbf{l} \in \mathcal{A}_{r+1}$, $\zeta := \gamma + \delta \mathbf{l} \in \mathcal{A}_{r+1}$.

The basis of $\mathcal{A}_r$ over $\mathbf{R}$ is denoted by $\mathbf{b}_r := \mathbf{b} := \{1, i_1, ..., i_{2^r-1}\}$, where $i_s^2 = -1$ for each $1 \le s \le 2^r - 1$, $i_{2^r} := \mathbf{l}$ is the additional element of the doubling procedure of $\mathcal{A}_{r+1}$ from the preceding Cayley-Dickson algebra $\mathcal{A}_r$. Their enumeration can be chosen as $i_{2^r+m} = i_m \mathbf{l}$ for each $m = 1, ..., 2^r - 1$, $i_0 := 1$. This implies that $\xi \mathbf{l} = \mathbf{l} \xi^*$ for each $\xi \in \mathcal{A}_r$, when $1 \le r$.

2. Remarks and notations.

The family of all $\mathcal{A}_r$ locally z-analytic functions $f(z)$ on a domain U in $\mathcal{A}_r$ with values in the Cayley-Dickson algebra $\mathcal{A}_r$ is denoted by $\mathcal{H}(U, \mathcal{A}_r)$ or $C^\omega_z(U, \mathcal{A}_r)$.

To rewrite a function from real variables z_j in the z-representation the following identities are used:

(1) $z_j = (-zi_j + i_j(2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2$

for each $j = 1, 2, ..., 2^r - 1$,

$$(2) \quad z_0 = (z + (2^r - 2)^{-1}\{-z + \sum_{k=1}^{2^r-1} i_k(zi_k^*)\})/2,$$

where $2 \le r \in \mathbf{N}$, z is a Cayley-Dickson number decomposed as

(3) $z = z_0 i_0 + ... + z_{2^r-1} i_{2^r-1} \in \mathcal{A}_r$, $z_j \in \mathbf{R}$ for each j, $i_k^* = \tilde{i}_k = -i_k$ for each $k > 0$, $i_0 = 1$, since $i_k(i_0 i_k^*) = i_0 = 1$, $i_k(i_j i_k^*) = -i_k(i_k^* i_j) = -(i_k i_k^*) i_j = -i_j$ for each $k \ge 1$ and $j \ge 1$ with $k \ne j$ (shortly $k \ne j \ge 1$), $i_k(i_k i_k^*) = i_k$ for each $k \ge 0$.

3. Notation. If $f : U \to \mathcal{A}_r$ is either z-differentiable or $\tilde{z}$-differentiable at $a \in U$ or on U, then we can write also $D_{\tilde{z}}$ instead of $\partial_{\tilde{z}} = \partial/\partial\tilde{z}$ and D_z instead of $\partial_z = \partial/\partial z$ at $a \in U$ or on U respectively in situations, when it can not cause a confusion, where U is an open domain in $\mathcal{A}_r$.

4. Proposition. *A function $f : U \to \mathcal{A}_r$ is z-differentiable at a point $a \in U$ if and only if F is Fréchet differentiable at a and $\partial_{\tilde{z}} f(z)|_{z=a} = 0$. If f is z-super-differentiable on U, then f is z-represented on U. A $(z, \tilde{z})$-differentiable function f at $a \in U$ is z-differentiable at $a \in U$ if and only if $D_{\tilde{z}} f(z, \tilde{z})|_{z=a} = 0$.*

Proof. For each canonical closed compact set U in $\mathcal{A}_r$ the set of all polynomial by z functions is dense in the space of all continuous on U Fréchet differentiable functions on $Int(U)$ relative to the compact-open topology due the generalization of Stone-Weierstrass' theorem over the Cayley-Dickson algebras (see also [10, 11]).

As usually a set A having structure of an $\mathbf{R}$-linear space and having distributive multiplications of its elements on Cayley-Dickson numbers $z \in \mathcal{A}_v$ from the left and from the right is called a left- and right-$\mathcal{A}_v$-module (or vector space over $\mathcal{A}_v$ if this terminology can not cause any confusion).

For two vector spaces A and B over $\mathcal{A}_v$ one can consider their ordered

tensor product $A \otimes B$ over $\mathcal{A}_v$ consisting of elements $a \otimes b := (a, b)$ such that $a \in A$ and $b \in B$, $\alpha(a, b) = (\alpha a, b)$ and $(a, b)\beta = (a, b\beta)$ for each $\alpha, \beta \in \mathcal{A}_v$, $(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2$ for each $a_1, a_2 \in A$ and $b_1, b_2 \in B$. In the aforementioned respect $A \otimes B$ is the $\mathbf{R}$-linear space and at the same time left and right module over $\mathcal{A}_v$. Then $A \otimes B$ has the structure of the vector space over $\mathcal{A}_v$. By induction consider tensor products $\{C_1 \otimes C_2 \otimes ... \otimes C_n\}_{q(n)}$, where $C_1, ..., C_n \in \{A, B\}$, $q(n)$ indicates on the order of tensor multiplications in the curled brackets $\{*\}$.

For two $\mathcal{A}_v$-vector spaces V and W their direct sum $V \oplus W$ is the $\mathcal{A}_v$-vector space consisting of all elements (a, b) with $a \in V$ and $b \in W$ such that $\alpha(a, b) = (\alpha a, \alpha b)$ and $(a, b)\beta = (a\beta, b\beta)$ for each α and $\beta \in \mathcal{A}_v$. Therefore, the direct sum of all different tensor products $\{C_1 \otimes C_2 \otimes ... \otimes C_n\}_{q(n)}$, which are $\mathbf{R}$-linear spaces and left and right modules over $\mathcal{A}_v$, provides the minimal tensor space $T(A, B)$ generated by A and B.

Operators ∂_z and $\partial_{\tilde{z}}$ are uniquely defined on $C^\omega_z(U, \mathcal{A}_r)$ and $C^\omega_{\tilde{z}}(U, \mathcal{A}_r)$ in terms of phrases, hence they are unique on the tensor space $T(C^\omega_z(U, \mathcal{A}_r), C^\omega_{\tilde{z}}(U, \mathcal{A}_r))$, which is dense in $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$, since $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r) := C^\omega{}_{{}_1z,\ {}_2z}(U^2, \mathcal{A}_r)|_{\ {}_1z=z,\ {}_2z=\tilde{z}}$. Therefore, operators ∂_z and $\partial_{\tilde{z}}$ are uniquely defined on $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$.

If there is a product fg of two phrases f and g from $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$, then if it is reduced to a minimal phrase ξ, then it is made with the help of $z^n z^m = z^{n+m}$ and $\tilde{z}^n \tilde{z}^m = \tilde{z}^{n+m}$ and identities for constants in $\mathcal{A}_r$, since no any shortening related with their permutation $z\tilde{z} = \tilde{z}z$ or substitution of z on $\tilde{z}$ or $\tilde{z}$ on z, for example, using the identity $\tilde{z} = l(zl^*)$ is not allowed in $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r)$ in accordance with our convention in §2.1 [10, 11], since $C^\omega_{z,\tilde{z}}(U, \mathcal{A}_r) := C^\omega{}_{{}_1z,\ {}_2z}(U^2, \mathcal{A}_r)|_{\ {}_1z=z,\ {}_2z=\tilde{z}}$ and in $C^\omega{}_{{}_1z,\ {}_2z}(U^2, \mathcal{A}_r)$ variables ${}_1z$ and ${}_2z$ do not commute, ${}_1z$ and ${}_2z$ are different variables which are not related. Therefore, $\partial_z \xi.h = (\partial_z f.h)g + f(\partial_z g.h)$ and $\partial_{\tilde{z}} \xi.h = (\partial_{\tilde{z}} f.h)g + f(\partial_{\tilde{z}} g.h)$, hence ∂_z and $\partial_{\tilde{z}}$ are correctly defined.

We consider the super-differentiability, but in accordance with our convention above for short we simply write differentiability over $\mathcal{A}_r$.

We can use a δ-approximation for each $\delta > 0$ of $Dg(z).h$ on a sufficiently small open subset V in U such that $z \in V$ and $|h| \leq 1$ by functions ζ_n polynomial in z and $\mathbf{R}$-homogeneous $\mathcal{A}_r$-additive in h, we also use the partition

of unity in U by C^ω_z-functions. Then consider functions ξ_n with ${\xi_n}'.h$ corresponding to ζ_n for each canonical closed compact subset W in U, since from each open covering of W we can choose a finite sub-covering of W.

Suppose that f is z-differentiable at a point a. The derivative $f'(z)$ is the $\mathbf{R}$-linear $\mathcal{A}_r$-additive operator, so to it an $\mathbf{R}$-linear operator on the Euclidean space $\mathbf{R^{2^rn}}$ corresponds (see §2.1, 2.2 [10, 11]). Then $f(a+h)-f(a) = \partial_a f(a).h+\epsilon(h)|h|$ and $\partial_{\tilde{z}} f(z)|_{z=a} = 0$, since generally for a $(z,\tilde{z})$-differentiable function $f(a+h)-f(a) = (\partial_a f(a)).h + (\partial_{\tilde{a}} f(a)).h + \epsilon(h)|h|$, where $\epsilon(h)$ is continuous by h and $\epsilon(0)=0$.

Vice versa, if F is Fréchet differentiable and $\partial_{\tilde{z}} f(z)|_{z=a} = 0$, then expressing $z_j i_j$ for each $j = 0,1,...,2^r-1$ through linear combinations of z with multiplication on constant coefficients from $\mathcal{A}_r$ on the left and on the right in accordance with Formulas $2(1-3)$ we get the increment of f as above.

The last statement of this proposition follows from Definition 2.2 [10, 11].

5. Proposition. *Let $g: U \to \mathcal{A}_r$, $r \ge 2$, and $f: W \to \mathcal{A}_r$ be two differentiable functions on U and W respectively such that $g(U) \subset W$, U and W are open in $\mathcal{A}_r$, where f and g are simultaneously either $(z,\tilde{z})$, or z, or $\tilde{z}$-differentiable. Then the composite function $f\circ g(z) := f(g(z))$ is differentiable on U and*

$$(Df\circ g(z)).h = (Df(g)).((Dg(z)).h)$$

for each $z \in U$ and each $h \in \mathcal{A}_r$, and hence $f\circ g$ is of the same type of differentiability as f and g.

Proof. Theorems 2.11, 2.15, 2.16, 3.10 and Corollary 2.13 [10, 11] establish the equivalence of notions of $\mathcal{A}_r$-holomorphic and $\mathcal{A}_r$ locally z-analytic classes of functions on open domains in $\mathcal{A}_r$.

In view of these results it is sufficient to prove this Proposition on open domains U and W, where $g(U) \subset W$ with $U = g^{-1}(W)$, if others conditions are the same, since z-differentiability is equivalent with the local z-analyticity (that is in the z-representation). Indeed, the composition of two locally z-analytic functions $f\circ g$ with such domains is the locally z-analytic function, analogously for $\tilde{z}$ and $(z,\tilde{z})$ differentiability.

Since g is differentiable, the function g is continuous and $g^{-1}(W)$ is open

in $\mathcal{A}_r$. In view of Proposition 4 above if f and g are simultaneously either z-differentiable or $\tilde{z}$-differentiable, then either $\partial_{\tilde{z}} f = 0$ and $\partial_{\tilde{z}} g = 0$ or $\partial_z f = 0$ and $\partial_z g = 0$ correspondingly on their domains.

Consider the increment of the composite function

$$f \circ g(z+h) - f \circ g(z) = (Df(g))|_{g=g(z)}.(g(z+h) - g(z)) + \epsilon_f(\eta)|\eta|,$$

where $\eta = g(z+h) - g(z)$, $g(z+h) - g(z) = (Dg(z)).h + \epsilon_g(h)|h|$ (see §2.2 [10, 11]). Since the derivative Df is $\mathcal{A}_r$-additive and $\mathbf{R}$-homogeneous (and continuous) operator on $\mathcal{A}_r$, we have
$f \circ g(z+h) - f \circ g(z) = (Df(g))|_{g=g(z)}.((Dg(z)).h) + \epsilon_{f\circ g}(h)|h|$, where

$$\epsilon_{f\circ g}(h)|h| := \epsilon_f((Dg(z)).h + \epsilon_g(h)|h|)|(Dg(z)).h + \epsilon_g(h)|h||$$

$$+[(Df(g))|_{g=g(z)}.(\epsilon_g(h))]|h|),$$

$$|(Dg(z)).h + \epsilon_g(h)|h|| \le [\|Dg(z)\| + |\epsilon_g(h)|]|h|, \text{ hence}$$

$$|\epsilon_{f\circ g}(h)| \le |\epsilon_f((Dg(z)).h+\epsilon_g(h)|h|)|[\|Dg(z)\|+|\epsilon_g(h)|]+\|(Df(g))|_{g=g(z)}\||\epsilon_g(h)|$$

and inevitably $\lim_{h\to 0} \epsilon_{f\circ g}(h) = 0$. Moreover, $\epsilon_{f\circ g}(h)$ is continuous in h, since ϵ_g and ϵ_f are continuous functions, Df and Dg are continuous operators. Evidently, if $\partial_{\tilde{z}} f = 0$ and $\partial_{\tilde{z}} g = 0$ on domains of f and g respectively, then $\partial_{\tilde{z}} f \circ g = 0$ on V, since $D = \partial_z + \partial_{\tilde{z}}$.

Suppose now that there are phrases corresponding to f_n and g_n denoted by μ_n and ν_n such that f_n and g_n uniformly converge to f and g respectively on each bounded canonical closed subset in W and U from a family $\mathcal{W}$ or $\mathcal{U}$ respectively, where $\mathcal{W}$ and $\mathcal{U}$ are coverings of W and U correspondingly. While $D_z\mu_n$ and $D_z\nu_n$ are fundamental sequences uniformly on each bounded canonical closed subset $P \in \mathcal{W}$ and $Q \in \mathcal{U}$ correspondingly relative to the operator norm (see Definitions in §2 [10, 11]). Then the sequence $\mu_n \circ \nu_n$ converges on each bounded canonical closed subset Q_1 such that $Q_1 \subset Q \cap g^{-1}(P)$, where $P \in \mathcal{W}$ and $Q \in \mathcal{U}$ in U, when n tends to the infinity. Moreover, $D_y\mu_n(y,\tilde{y}).(D_z\nu_n(z,\tilde{z}))|_{y=\nu_n(z,\tilde{z})}$ and $D_{\tilde{y}}\mu_n(y,\tilde{y}).(D_z\tilde{\nu}_n(z,\tilde{z}))|_{y=\nu_n(z,\tilde{z})}$ are the fundamental sequences of operators on each bounded canonical closed subset Q_1 so that $Q_1 \subset Q \cap g^{-1}(P)$, where $P \in \mathcal{W}$ and $Q \in \mathcal{U}$. The family $\mathcal{Q} := \{Q_1\}$ specified above evidently is the covering of V.

It remains to verify, that $D_z[\mu_n\circ\nu_n(z,\tilde{z})]=[(D_y\mu_n(y,\tilde{y}).(D_z\nu_n(z,\tilde{z}))+(D_{\tilde{y}}\mu_n(y,\tilde{y}).(D_z\tilde{\nu}_n(z,\tilde{z}))]|_{y=\nu_n(z,\tilde{z})}$.

Since the derivation operator D_z by z is $\mathbf{R}$-homogeneous and $\mathcal{A}_r$-additive, it is sufficient to verify this in the case $D_z[\eta\circ\psi(z,\tilde{z})]$ locally in balls, where both series uniformly converge. Here the phrase is written as:

$$(1)\quad \eta=\eta(z,\tilde{z})=\sum_k\{A_k,z,\tilde{z}\}_{q(k)}:=\{a_{k,1}\overbrace{z_{l_1}^{k_1}}...a_{k,p}\overbrace{z_{l_p}^{k_p}}\}_{q(k)},$$

$k=(k_1,...,k_p)$, $p=p(k)\in\mathbf{N}$, $0\le k_j\in\mathbf{Z}$ and $l_j\in\{1,2\}$ for each j, $a_{k,1},...,a_{k,p}\in\mathcal{A}_r$ are constants, $\overbrace{z_1}=z$, $\overbrace{z_2}=\tilde{z}$, z is the Cayley-Dickson variable, a vector q indicates on an order of the multiplication, $A_k=[a_{k,1},...,a_{k,p}]$, also

$$(2)\quad \psi=\psi(z,\tilde{z})=\sum_m\{B_m,z,\tilde{z}\}_{q(m)},$$

where $m=(m_1,...,m_s)$, $B_m=[b_{m,1},...,b_{m,s}]$, $b_{m,j}\in\mathcal{A}_r$ is a constant for each m,j, also $s\in\mathbf{N}$, $0\le m_j\in\mathbf{N}$ for each j. We have the identities

$$(3)\quad D_z\overbrace{z_l}=\delta_{l,1}\mathbf{1},\quad D_{\tilde{z}}\overbrace{z_l}=\delta_{l,2}\tilde{\mathbf{1}},$$

where $\mathbf{1}h:=h$ and $\tilde{\mathbf{1}}h:=\tilde{h}$ for each $h\in\mathcal{A}_r$. Using shifts $z\mapsto z-\zeta$ we can consider that the series are decomposed around a zero point. If a series η is uniformly converging on a canonical closed subset Q_1 as above, then

$$(4)\quad D_z\eta=\sum_k D_z\{a_{k,1}\overbrace{z_{l_1}^{k_1}}...a_{k,p}\overbrace{z_{l_p}^{k_p}}\}_{q(k)}$$

is also uniformly converging on Q_1 in accordance with our supposition about convergence of phrases and their derivatives.

In suitable Q_1 we deduce from Formula (4) that

$$(5)\quad D_z[\eta\circ\psi(z,\tilde{z})]=D_z\sum_k\{a_{k,1}\overbrace{\psi_{l_1}^{k_1}}...a_{k,p}\overbrace{\psi_{l_p}^{k_p}}\}_{q(k)}$$

$$=\sum_k D_z\{a_{k,1}\overbrace{\psi_{l_1}^{k_1}}...a_{k,p}\overbrace{\psi_{l_p}^{k_p}}\}_{q(k)}$$

$$=\sum_k[\{a_{k,1}(D_z\overbrace{\psi_{l_1}^{k_1}})...a_{k,p}\overbrace{\psi_{l_p}^{k_p}}\}_{q(k)}+...+\{a_{k,1}\overbrace{\psi_{l_1}^{k_1}}...a_{k,p}(D_z\overbrace{\psi_{l_p}^{k_p}})\}_{q(k)}]$$

$$=\sum_k\sum_{{}_1m,...,{}_pm;p=p(k)}[\{a_{k,1}(D_z(\overbrace{\{B_{{}_1m},z,\tilde{z}\}_{q({}_1m)})_{l_1}^{k_1}})...a_{k,p}(\overbrace{\{B_{{}_pm},z,\tilde{z}\}_{q({}_pm)})_{l_p}^{k_p}}\}_{q(k)}$$

$$+...+\{a_{k,1}(\overbrace{\{B_{\,{}_1m}, z, \tilde{z}\}_{q(\,{}_1m)}})^{k_1}_{l_1}...a_{k,p}(D_z(\overbrace{\{B_{\,{}_pm}, z, \tilde{z}\}_{q(\,{}_pm)}})^{k_p}_{l_p})\}_{q(k)}.$$

On the other hand, we have

$$D_z\{B_m, z, \tilde{z}\}_{q(m)} = \{b_{m,1}(D_z \overbrace{z^{m_1}_{l_1}})...b_{m,s} \overbrace{z^{k_s}_{l_s}}\}_{q(m)}+...+\{b_{m,1} \overbrace{z^{m_1}_{l_1}}...b_{m,s}(D_z \overbrace{z^{k_s}_{l_s}})\}_{q(k)}$$

due to the Leibnitz rule, hence

$$D_z \overbrace{\psi^p} = \sum_{{}_1m,...,\,{}_pm} [(...(D_z \overbrace{\{B_{\,{}_1m}, z, \tilde{z}\}_{q(\,{}_1m)}})...) \overbrace{\{B_{\,{}_pm}, z, \tilde{z}\}_{q(\,{}_pm)}}$$

$$+...+(...(\overbrace{\{B_{\,{}_1m}, z, \tilde{z}\}_{q(\,{}_1m)}} \overbrace{\{B_{\,{}_2m}, z, \tilde{z}\}_{q(\,{}_2m)}})...)(D_z \overbrace{\{B_{\,{}_pm}, z, \tilde{z}\}_{q(\,{}_pm)}})]$$

$$= (D_y \overbrace{y^p}).(D_z\psi) + (D_{\tilde{y}} \overbrace{y^p}).(D_z\tilde{\psi})$$

due to Formulas (3), where $y = \psi(z, \tilde{z})$. Thus Formulas $(5, 6)$ imply that

(7) $\quad D_z(\eta\circ\psi(z, \tilde{z})) = [(D_y\eta(y, \tilde{y})).(D_z\psi(z, \tilde{z}))+(D_{\tilde{y}}\eta(y, \tilde{y})).(D_z\tilde{\psi}(z, \tilde{z}))]|_{y=\psi(z,\tilde{z})}.$

Quite analogously or using the conjugation one deduces that

(8) $\quad D_{\tilde{z}}(\eta\circ\psi(z, \tilde{z})) = [(D_{\tilde{y}}\eta(y, \tilde{y})).(D_{\tilde{z}}\psi(z, \tilde{z}))+(D_y\eta(y, \tilde{y})).(D_{\tilde{z}}\tilde{\psi}(z, \tilde{z}))]|_{y=\psi(z,\tilde{z})}.$

Particularly, we get

(9) $\quad D_z(\eta\circ\psi(z)) = (D_y\eta(y)).(D_z\psi(z))|_{y=\psi(z)},$ when $D_{\tilde{z}}\eta(z, \tilde{z}) = 0$ and $D_{\tilde{z}}\psi(z, \tilde{z}) = 0$

and

(10) $\quad D_{\tilde{z}}(\eta\circ\psi(\tilde{z})) = (D_{\tilde{y}}\eta(\tilde{y})).(D_{\tilde{z}}\psi(\tilde{z}))|_{y=\psi(\tilde{z})},$ when $D_z\eta(z, \tilde{z}) = 0$ and $D_z\psi(z, \tilde{z}) = 0.$

Combining Formulas $(7, 8)$ and taking into account the proof given above and applying Proposition 2.3 [10, 11] we infer the chain rule, when both η and ψ are either the z or $\tilde{z}$ or $(z, \tilde{z})$-differentiable, that is in all three considered cases.

6. Some elementary functions and their non-commutative Riemann surfaces.

In this article some elementary facts about analytic functions z^n, $z^{1/n}$, $\exp(z)$ and $Ln(z)$ of the Cayley-Dickson variables are used. They were considered in details in previous works [11, 10, 12]. Recall that the exponential function is defined by the power series

$$\exp(z) := 1 + \sum_{n=1}^{\infty} z^n/n!$$

converging on the entire Cayley-Dickson algebra $\mathcal{A}_r$. It has the periodicity property $\exp(M(\phi + 2\pi k)) = \exp(M\phi)$ for each purely imaginary Cayley-Dickson number M of the unit norm $|M| = 1$ for any real number $\phi \in \mathbf{R}$ and each integer number $k \in \mathbf{Z}$. The restriction of such exponential function on each complex plane $\mathbf{C}_M := \mathbf{R} \oplus M\mathbf{R}$ coincides with the traditional complex exponential function. Since the inverse function $Ln(z)$ of $z = \exp(x)$ is defined on every complex plane $\mathbf{C}_M \setminus \{0\}$ with the pricked zero point, the logarithmic function $Ln(z)$ is defined on $\mathcal{A}_r \setminus \{0\}$. This logarithmic function is certainly multi-valued. Consider the bunch of complex planes $\mathbf{C}_M$ intersecting by the real line $\mathbf{R}i_0$ as the geometric realization of $\mathcal{A}_r$. Certainly $\mathbf{C}_M = \mathbf{C}_{-M}$, so we take the set

$\mathbf{S}_r^+ := \{M \in \mathcal{A}_r : |M| = 1,\ M = M_1 i_1 + ... + M_{2^r-1} i_{2^r-1}$, either $M_1 > 0$, or $M_1 = 0$ and $M_2 > 0$, or..., or $M_1 = ... = M_{2^r-2} = 0$ and $M_{2^r-1} > 0\}$, where $M_1, ..., M_{2^r-1} \in \mathbf{R}$, $2 \le r$. Then $\mathcal{A}_r = \bigcup_{M \in \mathbf{S}_r^+} \mathbf{C}_M$.

If A and B are two subsets in a complete uniform space X and $\theta : A \to B$ is a continuous bijective mapping, the equivalence relation $a \Upsilon b$ by definition means $b = \theta(a)$ for $a \in A$ and $b \in B$; or $b \Upsilon a$ means $a = \theta^{-1}(b)$. When θ is uniformly continuous, θ has a uniformly continuous extension $\theta : cl(A) \to cl(B)$, where $cl(A)$ denotes the closure of the set A in X (see Theorem 8.3.10 in [3]). Certainly, the mapping θ can be specified by its graph $\{(x_1, x_2) :\ x_2 = \theta(x_1),\ x_1 \in A\}$. We say, that A and B are glued (by θ), if $B = \theta(A)$ and the natural quotient mapping $\pi : X \to X/\Upsilon$ is given, where X/Υ denotes the quotient space (see §2.4 [3]). For the uniformly continuous θ this means that the gluing is extended from A onto $cl(A)$.

In the complex case to construct the Riemann surface of the logarithmic function one takes traditionally the complex plane $\mathbf{C}$ cut by the set $Q_1 := \{z = x + iy \in \mathbf{C} :\ x < 0\}$ and marking two respective points x_1 and x_2 of two edges $Q_{1,1}$ and $Q_{1,2}$ of the cut Q_1 arising from each given point $x < 0$, where $i = i_1$. Then one embeds $\mathbf{C}$ into either $\mathbf{C} \times \mathbf{R}$ or $\mathbf{C} \times \mathbf{R}i$ and bents the obtained surface slightly along the perpendicular axis e_3 to $\mathbf{C}$ by neighborhoods of two edges of the cut Q_1 and gets the new surface $\mathcal{C}$. Taking the countable infinite family of such surfaces $\mathcal{C}^j$ with the edges of cuts $Q^j_{1,1}$ and $Q^j_{1,2}$ and gluing by respective points of the cuts $Q^j_{1,2}$ with $Q^{j+1}_{1,1}$ for each j

one gets the Riemann surface of the logarithmic function, where $j \in \mathbf{Z}$ (see, for example, [9]).

Analogous procedure to construct the Riemann surface is in the cases $r \geq 2$: one cuts $\mathcal{A}_r$ by Q_r and gets two edges $Q_{r,1}$ and $Q_{r,2}$ of the cut. This is described below.

If K and $M = KL$ are two purely imaginary Cayley-Dickson numbers with $|K| = |M| = |L| = 1$ so that they are orthogonal $K \perp M$, that is $Re(KM) = 0$, then $(K\mathbf{R}) \oplus (M\mathbf{R}) = K(\mathbf{R} \oplus L\mathbf{R})$ and L is also purely imaginary. Consider any path $\gamma : [0,1] \to K\mathbf{R} \oplus M\mathbf{R}$ winding one time around zero such that $\gamma(t) \neq 0$ for each t. Each $z \in K\mathbf{R} \oplus M\mathbf{R}$ can be written in the polar form $z = |z|Ke^{L\phi} = |z|\exp(\pi K e^{L\phi}/2)$, where $\phi = \phi(z) \in \mathbf{R}$, since $Ke^{L\phi} = K\cos(\phi) + (KL)\sin(\phi)$ due to Euler's formula, hence $|Ke^{L\phi(\gamma(t))}| = 1$ for each t (see Section 3 in [11, 10]). In particular, $\gamma(t) = |\gamma(t)|\exp(\pi K e^{L\phi(t)}/2)$. But $Ke^{L\phi(\gamma(0))} = Ke^{L\phi(\gamma(1))}$ such that the logarithm $Ln\,\gamma(t) = Ln\,|\gamma(t)| + \pi K e^{L\phi(\gamma(t))}/2$ does not change its branch, when the path $\gamma(t) \in (K\mathbf{R}) \oplus (M\mathbf{R}) \setminus \{0\}$ winds around zero, since $|\pi K e^{L\phi}/2| = \pi/2 < \pi$. Due to the homotopy theorem (see [11, 10]) this means that the logarithm $Ln\,\gamma(t)$ does not change its branch, when $Re(\gamma(t)) = 0$ for each t for the path γ winding around zero with $|\gamma(t)| > 0$ for each t.

The first simple construction for $r \geq 2$ is the following. Take the set $Q_r := (-\infty, 0)\mathbf{S}_r^+ := \{z = tx : \ t \in (-\infty, 0), x \in \mathbf{S}_r^+\}$ and cut $\mathcal{A}_r$ by Q_r. The set Q_r is the union $Q_r = \bigcup_{j=1}^{2^r-1} \Omega_{j,r}$ of subsets $\Omega_{j,r} := \{z \in Q_r : \ z_0 = 0, ..., z_j = 0\}$ so that $\Omega_{j,r}$ is contained in the boundary of the preceding set $\Omega_{j,r} \subset \partial\Omega_{j-1,r}$ for each $j = 1, ..., 2^r - 1$, $dim\ \Omega_{j,r} = dim\ \Omega_{j-1,r} - 1$, moreover, $\mathbf{R}\mathbf{S}_r^+ = \mathcal{I}_r := \{z \in \mathcal{A}_r : \ Re(z) = 0\}$. Therefore, from each point $z \in Q_r$ two and only two different points z_1 and z_2 arise while cutting of $\mathcal{A}_r$ by Q_r.

It is useful to embed $\mathcal{A}_r$ either into $\mathcal{A}_r \times \mathbf{R}^{2^r-1}$ or into $\mathcal{A}_r \times \mathcal{I}_r$. Then one marks all pairs of respective points z_1 and z_2 arising from $z \in Q_r$ after cutting, slightly bents the cut copy of $\mathcal{A}_r \setminus \{0\}$ by $(2^r - 1)$ axes perpendicular to $\mathcal{A}_r$ by two neighborhoods of two edges of the cut Q_r. Thus one gets the 2^r dimensional surface $\mathcal{C}_r$ with two edges $Q_{r,1}$ and $Q_{r,2}$ of the cut. Taking the countable infinite family of such surfaces $\mathcal{C}_r^j$ with edges of the cuts $Q_{r,1}^j$ and $Q_{r,2}^j$, $j \in \mathbf{Z}$, and gluing respective points of edges $Q_{r,2}^j$ with $Q_{r,1}^{j+1}$ for

each j one gets the Riemann surface $\mathcal{R}_r = \mathcal{R}_{r,Ln}$ of the logarithmic function $Ln(z) : \mathcal{A}_r \setminus \{0\} \to \mathcal{R}_r$. Thus the latter mapping is already univalent with the image in the Riemann surface (see in details [11, 10, 12]).

For convenience we attach numbers 1 and 2 to faces in such manner that the winding around zero in the complex plane $\mathbf{C}_M$ embedded into $\mathcal{R}_r$ counterclockwise means the transition through the cut from $Q^j_{r,2}$ to $Q^j_{r,1}$ for each M in the connected set $\mathbf{S}^+_r$.

For the function $z^{1/n}$ with $n \in \mathbf{N}$ its Riemann surface $\mathcal{R}_{r,z^{1/n}}$ is obtained from n copies of surfaces $\mathcal{C}^j_r$, $j = 1, ..., n$, by gluing the corresponding points of edges $Q^j_{r,2}$ with $Q^{j+1}_{r,1}$ for $j = 1, ..., n-1$ and of $Q^n_{r,2}$ with $Q^1_{r,1}$ (see also [9, 11, 10, 12]).

Another more complicated construction is described below. Now we take the set

$Q_r := \bigcup_{j=1}^{2^r-1} P_j$, where

$P_j := \{z \in \mathcal{A}_r : z = z_0 i_0 + ... + z_{2^r-1} i_{2^r-1};\ z_0 < 0 \text{ and } z_j = 0\}$, where $z_0, ..., z_{2^r-1} \in \mathbf{R}$, $2 \le r$.

Let $z = z_0 + z'$ be the Cayley-Dickson number with the negative real part $z_0 < 0$ and the imaginary part $Im(z) = z'$, which can be written in the form $z' = |z'| \exp(\pi K e^{L\phi(z')}/2)$ and $z = |z| e^{P\psi}$, where K, L and P are purely imaginary Cayley-Dickson numbers of the unit norm, ϕ and $\psi \in \mathbf{R}$ are reals, $Re(KL^*) = 0$, $|KL| = 1$. This gives the relation $\cos(\psi) = z_0/|z|$ so that the parameter ψ is in the interval $\pi/2 + 2\pi k < \psi < 3\pi/2 + 2\pi k$ for some integer number k. This means that for a continuous path γ contained in the set Q_r the parameter $\psi(\gamma(t))$ is the continuous function of the real variable $t \in \mathbf{R}$ and remains in the same interval $(\pi/2 + 2\pi k, 3\pi/2 + 2\pi k)$, consequently, the logarithmic function $Ln\gamma(t)$ preserves its branch along such path $\gamma(t)$. This shows that after the first cut along Q_r the obtained sets $Q_{r,1}$ and $Q_{r,2}$ need not be further cut. Thus the described reason simplifies the construction of the Riemann surface.

Each continuous path $\gamma : [0,1] \to \mathcal{A}_r$ can be decomposed as the point-wise sum and as the composition (join) up to the homotopy satisfying the conditions of the homotopy theorem [11, 10] of paths $\gamma_{k,l}$ in the planes $(\mathbf{R}i_k) \oplus (\mathbf{R}i_l)$ for each $k < l \in \Lambda$ for the corresponding subset $\Lambda \subset \{0, 1, ..., 2^r - 1\}$.

If $|\gamma(t)| > 0$ for each t we take $\gamma_{k,l}$ with $|\gamma_{k,l}(t)| > 0$ on $[0,1]$ for all $k < l \in \Lambda$. When $\gamma[0,1]$ does not intersect Q_r one can choose $\gamma_{k,l}$ with images $\gamma_{k,l}[0,1]$ also non-intersecting with Q_r for all $k < l \in \Lambda$. Due to the homotopy theorem the logarithm $Ln\ \gamma(t)$ does not change its branch along such continuous path $\gamma(t)$, since this is the case for $Ln\ \gamma_{k,l}(t)$ for all $k < l \in \Lambda$.

For each $z \in P_j \setminus \bigcup_{m,m\ne j} P_m$ cutting by Q_r gives two points. If $z \in (P_k \cap P_j) \setminus \bigcup_{m,m\ne k,m\ne j} P_m$ with $k < j$ cutting gives four points which can be organized into respective pairs after cutting of P_k and then of P_j. This procedure gives pairs $(z_{1,1}; z_{2,2})$ and $(z_{1,2}; z_{2,1})$. For $z \in (P_{k_1} \cap ... \cap P_{k_l}) \setminus \bigcup_{m,m\ne k_1,...,m\ne k_l} P_m$ consider pairs of points appearing from the preceding point after cutting of P_{k_j}, $j = 1, ..., l$ by induction, where $k_1 < ... < k_l$. One can do it by induction by all $l = 1, ..., 2^r - 1$ and all possible subsets $1 \le k_1 < k_2 < ... < k_l \le 2^r - 1$. It produces points denoted by $z_{a_1,...,a_l}$. Here $a_j = 1$ corresponds to the face of the cut indicated by the condition $z_j \le 0$, while $a_j = 2$ corresponds to the face of the cut with $z_j \ge 0$. Two points $z_{a_1,...,a_l}$ and $z_{b_1,...,b_l}$ form the pair of respective points, when $a_j + b_j = 3$ for each j, where $l \in \{1, ..., 2^r - 1\}$ is the index of such points. To each point $z \in (P_{k_1} \cap ... \cap P_{k_l}) \setminus \bigcup_{m,m\ne k_1,...,m\ne k_l} P_m$ of index $l \ge 1$ the point $M \in \mathbf{S}_r^+$ corresponds such that $M = M_1 i_1 + ... + M_{2^r-1} i_{2^r-1}$ and $z \in \mathbf{C}_M$, where $M_{k_1} = 0$,...,$M_{k_l} = 0$.

Then as above after cutting of $\mathcal{A}_r$ by Q_r one gets the 2^r dimensional surface $\mathcal{C}_r$ with two edges $Q_{r,1}$ and $Q_{r,2}$ of the cut. For the countable infinite family of the surfaces $\mathcal{C}_r^j$ with edges of the cuts $Q_{r,1}^j$ and $Q_{r,2}^j$, $j \in \mathbf{Z}$, one glues respective points of edges $Q_{r,2}^j$ with $Q_{r,1}^{j+1}$ for each j. Thus one gets the Riemann surface $\mathcal{R}_r = \mathcal{R}_{r,Ln}$ of the logarithmic function $Ln(z) : \mathcal{A}_r \setminus \{0\} \to \mathcal{R}_r$. Therefore, the latter mapping is univalent with the image in the Riemann surface (see in details [11, 10, 12]). For simplicity one can glue at first pairs of respective points of index $l = 1$ and extend gluing by continuity on points of index $l > 1$ (see above). The first construction for $r \ge 2$ operates with points of index one only. That is why it is simpler than the second procedure.

The reader can lightly see that geometrically $\mathcal{R}_{r,f}$ (for both types described above) is the bunch $\bigcup\{\mathcal{R}_{1,f,M} : M \in \mathbf{S}_r^+\}$ of complex Riemann surfaces $\mathcal{R}_{1,f,M}$ for the restrictions $f|_{\mathbf{C}_M}$ on the complex planes $\mathbf{C}_M$ of the func-

tion f, where either $f = Ln$ or $f = z^{1/n}$ among those functions considered here.

The main problem of the multi-dimensional geometry is in depicting its objects on the two-dimensional sheet of paper so one uses either projections or sections of the multi-dimensional surfaces. The bunch interpretation $\bigcup_{M\in \mathbf{S}_r^+} \mathcal{R}_{1,f,M}$ helps to imagine how the 2^r-dimensional Riemann surface $\mathcal{R}_{r,f}$ is organized for $f = Ln$ and $f = z^{1/n}$ with $n \in \mathbf{N}$.

7. In this section a special meromorphic function of Cayley-Dickson variables is constructed and its properties are used to demonstrate the following theorem.

Fermat's last theorem. *No three positive integers a, b and c can satisfy the equation $a^n + b^n = c^n$ for any integer value of n greater than two.*

Proof. We take the Cayley-Dickson algebra $\mathcal{A}_v$ with a sufficiently large index $v \ge 3$ (see below) and choose three different $\mathbf{R}$-linear embeddings of the quaternion skew field $\mathbf{H} = \mathcal{A}_2$ into it. For this we take the standard basis of generators $\{i_0, ..., i_{2^v-1}\}$ of $\mathcal{A}_v$. These embeddings use the doubling generators $\{i_1, i_2\}$ for the first; $\{i_4, i_8\}$ for the second and $\{i_{16}, i_{32}\}$ for the third embedding correspondingly and their different products, since these purely imaginary pairs $\{M, K\}$ satisfy the following conditions $Re(MK) = 0$ and $MK = -KM$ and $|MK| = |M||K|$, $|M| = |K| = 1$ (see also [1, 2, 7]).

The first embedding $\theta_1 : \mathbf{H} \to \mathcal{A}_r$ is the usual identical mapping $\theta_1(i_k) = i_k$ for $k = 0, 1, 2, 3$. We certainly have $\theta_l(x) = x$ for each real number x and $l = 1, 2, 3$. The second embedding θ_2 corresponds to $\mathbf{H}$ written as $\bigoplus_{k=0}^3 \mathbf{R} i_{4k}$, where we use enumeration $i_{2^r+p} = i_p i_{2^r}$ for $r \ge 1$ and $p = 1, ..., 2^r - 1$. The third embedding θ_3 corresponds to $\mathbf{H}$ with the basis of generators $\{i_{16k} : k = 0, ..., 3\}$, i.e. $\theta_3(\mathbf{H}) = \bigoplus_{k=0}^3 \mathbf{R} i_{16k}$.

In three copies $\mathbf{H}_l := \theta_l(\mathbf{H})$, $l = 1, 2, 3$ of the quaternion skew field $\mathbf{H}$ embedded into the Cayley-Dickson algebra $\mathcal{A}_v$ we consider contained in them copies of the complex field $\mathbf{C}_l = \theta_l(\mathbf{C}) = \bigoplus_{k=0}^1 \mathbf{R} i_{2^{2(l-1)}k}$. Then we take the doubling generators i_{64} and i_{128} and i_{256} such that $(\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 + \mathbf{H}_2\mathbf{H}_3)$ and $\mathbf{H}_1 i_{64}$ and $\mathbf{H}_2 i_{128}$ and $\mathbf{H}_3 i_{256}$ do not intersect pairwise, where $(\mathbf{H}_1 + \mathbf{H}_2 + \mathbf{H}_3 + \mathbf{H}_2\mathbf{H}_3) = \{z \in \mathcal{A}_v : \ z = {}_1z + {}_2z + {}_3z + {}_4z, \ {}_lz \in \mathbf{H}_l, \ l = 1, 2, 3, \ {}_4z \in \mathbf{H}_2\mathbf{H}_3\}$, $AB := \{z \in \mathcal{A}_v : \ z = xy, \ x \in A, \ y \in B\}$ for subsets A and $B \subset \mathcal{A}_v$.

Variables in them we denote by $w \in \mathbf{C}_1$, $x \in \mathbf{C}_2$ and $y \in \mathbf{C}_3$. That is $wi_{64} + xi_{128} + yi_{256} \in \mathcal{A}_v$ for each w, x, y. Therefore, it is sufficient to take $v = 9$.

Two Cayley-Dickson numbers p and s are orthogonal $p \perp s$ by the definition if an only if $Re(ps^*) = 0$, because

(1) $Re(ps^*) = \sum_{j=0}^{2^v-1} p_j s_j$.

Next we take $\mathbf{R}$-linear projection mappings. For each $j = 0, ..., 2^v - 1$ the $\mathbf{R}$-linear projection operator $\pi_j : \mathcal{A}_v \to \mathbf{R} i_j$ exists due to Formulas $2(1-3)$ so that $\pi_j(z) = i_j z_j = z_j i_j$:

(2) $\pi_j(z) = (-i_j(zi_j) - (2^v-2)^{-1}\{-z + \sum_{k=1}^{2^v-1} i_k(zi_k^*)\})/2$

for each $j = 1, 2, ..., 2^v - 1$,

$$(3) \quad \pi_0(z) = (z + (2^v-2)^{-1}\{-z + \sum_{k=1}^{2^v-1} i_k(zi_k^*)\})/2,$$

where $2 \le v \in \mathbf{N}$. Here we take $v = 9$.

Combining into suitable sums these projection operators one gets $\mathbf{R}$-linear projection operators $\upsilon_l : \mathcal{A}_9 \to \mathbf{C}_l i_{\kappa(l)}$, where $\kappa(1) = 64$, $\kappa(2) = 128$ and $\kappa(3) = 256$, $\upsilon_1 = \sum_{j=0}^1 \pi_{j+64}$, $\upsilon_2 = \sum_{j=0}^1 \pi_{4j+128}$, $\upsilon_3 = \sum_{j=0}^1 \pi_{16j+256}$. The latter projection operators induce $\mathbf{R}$-linear operators $\tau_l : \mathcal{A}_v \to \mathbf{C}_l$ so that $\tau_l(z) = \upsilon_l(z) i_{\kappa(l)}^*$ for each $z \in \mathcal{A}_v$ and $l = 1, 2, 3$, particularly, $\tau_1(wi_{\kappa(1)} + z) = w$ for each $z \perp wi_{\kappa(1)}$, $\tau_2(z + xi_{\kappa(2)}) = x$ for each $z \perp xi_{\kappa(2)}$ and $\tau_3(z + yi_{\kappa(3)}) = y$ for each $z \perp yi_{\kappa(3)}$.

In accordance with Formulas $2(1-3)$ and $(2, 3)$ each mapping $\tau_l(z)$ has the finite phrase expression of the type

(4) $\tau_l(z) = \sum_m (\alpha_{l,m}(z\beta_{l,m})) i_{\kappa(l)}^*$

in the z-representation, which we fix, where $z \in \mathcal{A}_v$, $\alpha_{l,m}$ and $\beta_{l,m} \in \mathcal{A}_v$ are Cayley-Dickson constants.

With the help of these $\mathbf{R}$-linear mappings τ_l we define the function

(5) $g(z) := \exp[\tau_1(z) \exp(2\pi L_1 \tau_1^{1/n}(z))]$

$-(\exp[\tau_2(z) \exp(2\pi L_2 \tau_2^{1/n}(z))])(\exp[\tau_3(z) \exp(2\pi L_3 \tau_3^{1/n}(z))]) + |z - \sum_{l=1}^3 \upsilon_l(z)|^2 L_4$

on the Cayley-Dickson algebra $\mathcal{A}_v$, where $L_1 = i_2$, $L_2 = i_8$ and $L_3 = i_{32}$, $L_4 = i_{511}$, while $n \ge 3$ is a natural number, $|z|^2 = zz^*$ and $|z| = \sqrt{zz^*}$, $|z|^2 = \sum_j z_j^2$, $z = \sum_j z_j i_j$, $z_j \in \mathbf{R}$ for each j, $z \in \mathcal{A}_v$. We take the branch of the square root function $\sqrt{z}$ so that $\sqrt{b} > 0$ for each $b > 0$. The con-

jugated number we write in the z-representation as $z^* = 2\pi_0(z) - z$, where $\pi_0(z) = z_0$ is given by Formula (3). That is, the function $|z|$ is written in the z-representation.

Since the exponent series

(6) $e^z = \sum_{m=0}^{\infty} z^m/m!$

converges for each $z \in \mathcal{A}_v$ and each function $\tau_l(z)$ is written in the z-representation, this function $g(z)$ is analytic in the z-representation on the Cayley-Dickson algebra $\mathcal{A}_v$.

Each twice iterated exponent can be written in the form

$$(7) \quad \exp[\tau_l(z)\exp(2\pi L_l \tau_l^{1/n}(z))] = \exp[Re\{\tau_l(z)\exp(2\pi L_l \tau_l^{1/n}(z))\}][\cos(|\tau_l(z)|)$$
$$+\frac{\sin(|\tau_l(z)|)}{|\tau_l(z)|} Im\{\tau_l(z)\exp(2\pi L_l \tau_l^{1/n}(z))\}] =: e_l(z)$$

for $\tau_l(z) \neq 0$, while $e^0 = 1$ for $\tau_l(z) = 0$. Indeed, the norm in the quaternion skew field $\mathbf{H}_l$ is multiplicative and the quaternion skew field is without divisors of zero. Moreover, $L_l \perp \tau_l^{1/n}(z)$ and the product $2\pi L_l \tau_l^{1/n}(z)$ is purely imaginary and hence $|\exp(2\pi L_l \tau_l^{1/n}(z))| = 1$, where $Im(z) = z - Re(z)$ is the imaginary part of a Cayley-Dickson number z, $Re(z) = (z + z^*)/2$ is the real part of z. Its value $e_l(z)$ belongs to the embedded copy $\mathbf{H}_l$ of the quaternion skew field, $\mathbf{H}_l \hookrightarrow \mathcal{A}_v$.

On the other hand, the intersection of three embedded copies of the quaternion skew field $\mathbf{H}_1 \cap \mathbf{H}_2 \cap \mathbf{H}_3 = \mathbf{R}$ is equal to the real field, which is the center $Z(\mathcal{A}_v) = \mathbf{R}$ of the Cayley-Dickson algebra $\mathcal{A}_v$, also $\mathbf{H}_1 \cap (\mathbf{H}_2\mathbf{H}_3) = \mathbf{R}$. The last term $|z - \sum_{l=1}^{3} v_l(z)|^2 L_4$ in Formula (5) is orthogonal to other additives $\exp[\tau_1(z)\exp(2\pi L_1 \tau_1^{1/n}(z))]$ and
$-(\exp[\tau_2(z)\exp(2\pi L_2 \tau_2^{1/n}(z))])(\exp[\tau_3(z)\exp(2\pi L_3 \tau_3^{1/n}(z))])$ there. Therefore, $g(z)$ may be equal to zero only when all three twice iterated exponents have real values.

Each number in the complex field $p \in \mathbf{C}_l$ is orthogonal to the doubling generator L_l of the quaternion skew field $\mathbf{H}_l$ and in accordance with Formula (7) the iterated exponent $e_l(z)$ is real only when $|\tau_l^{1/n}(z)| \in \mathbf{Z}/2 := \{0, \pm 1/2, \pm 1, \pm 3/2, ...\}$, since $Re(L_l \tau_l^{1/n}(z)) = 0$ and

(8) $\exp(2\pi L_l \tau_l^{1/n}(z)) = \cos(2\pi |\tau_l^{1/n}(z)|) + \frac{\sin(2\pi |\tau_l^{1/n}(z)|)}{|\tau_l^{1/n}(z)|} L_l \tau_l^{1/n}(z)$.

The number $\tau_l(z) \in \mathbf{C}_l$ is complex, consequently, $\tau_l^{1/n}(z)$ has n distinct

isolated roots in the complex field $\mathbf{C}_l$ corresponding to n branches of the function $z^{1/n}$. Take for definiteness the branch of $g(z)$ corresponding to the branch of the n-th root such that $b^{1/n} > 0$ for each $b > 0$.

In accordance with the notation above

(9) $g(z) = \exp[w\exp(2\pi L_1 w^{1/n})] - (\exp[x\exp(2\pi L_2 x^{1/n})])(\exp[y\exp(2\pi L_3 y^{1/n})]) + |z - \sum_{l=1}^3 \upsilon_l(z)|^2 L_4 =: \psi(w,x,y; z - \sum_{l=1}^3 \upsilon_l(z))$,
where $w = \tau_1(z)$, $x = \tau_2(z)$ and $y = \tau_3(z)$.

This function $\psi(w,x,y; z - \sum_{l=1}^3 \upsilon_l(z))$ may have zeros only when all three complex variables $w_1 := w \in \mathbf{C}_1$, $w_2 := x \in \mathbf{C}_2$ and $w_3 := y \in \mathbf{C}_3$ are real n-th powers of half-integer or integer numbers $t_l \in \mathbf{Z}/2$, $w_l = t_l^n$, while $z = \sum_{l=1}^3 \upsilon_l(z)$, since $\frac{\sin(|t_l^n|)}{|t_l^n|} \ne 0$ is non-zero for each non-zero $t_l \in (\mathbf{Z}/2)\setminus\{0\}$, while $\lim_{\phi\to 0} \frac{\sin(\phi)}{\phi} = 1$ (see also Formulas $(7-9)$ above). It can be lightly seen that $\psi(w,x,y; z - \sum_{l=1}^3 \upsilon_l(z)) = 0$ for $w = 0$ and $x = -y = t_2^n$ with $t_2 \in \mathbf{Z}/2$ for n odd, also $x = 0$ and $w = y = t_1^n$ with $t_1 \in \mathbf{Z}/2$, also $y = 0$ and $w = x = t_1^n$ with $t_1 \in \mathbf{Z}/2$, when $z = \sum_{l=1}^3 \upsilon_l(z)$. That is, the function $g(z)$ has only isolated zeros in the Cayley-Dickson algebra $\mathcal{A}_v$. It is shown below that this function has not any zeros when the product of arguments $wxy \ne 0$ is non-zero for $n \ge 3$.

The equality $g(z) = 0$ is equivalent to

(10) $s(t_1)t_1^n = s(t_2)t_2^n + s(t_3)t_3^n$ and $z = \sum_{l=1}^3 \upsilon_l(z)$,

where $t_1, t_2, t_3 \in \mathbf{Z}/2$, $s(t_l) = 1$ for an integer $t_l \in \mathbf{Z}$ and $s(t_l) = -1$ for non-integer half-integer $t_l \in \mathbf{Z} + 1/2$. If one of these numbers is non-integer half-integer: either $t_1 = k + 1/2$ or $t_2 = l + 1/2$ or $t_3 = m + 1/2$, where $k, l, m \in \mathbf{Z}$, then Equality (10) is equivalent to $s(t_1)(2k+1)^n = s(t_2)(2t_2)^n + s(t_3)(2t_3)^n$ or $s(t_1)(2t_1)^n = s(t_2)(2l+1)^n + s(t_3)(2t_3)^n$ or $s(t_1)(2t_1)^n = s(t_2)(2t_2)^n + s(t_3)(2m+1)^n$ respectively. That is Equality (10) is always equivalent to the corresponding equality for integers.

If the number n is even and (10) is satisfied, then it is also satisfied for $(\pm t_1, \pm t_2, \pm t_3)$. If n is odd and Equality (10) is satisfied, then it is also satisfied for the triple $(-t_1, -t_2, -t_3)$.

Considering different possible signs $s(t_1)$, $s(t_2)$, $s(t_3)$, $sign(t_1)$, $sign(t_2)$ and $sign(t_3)$ one leads to the conclusion that (10) is equivalent to the equality

(11) $a^n + b^n = c^n$

for non-negative integers, where $c = \nu \max(|t_1|, |t_2|, |t_3|)$; $\nu = 1$, when t_1 and t_2 and t_3 are integer; $\nu = 2$, when at least one of these numbers t_1 or t_2 or t_3 is non-integer half-integer. Indeed, for non-negative numbers t_1 and t_2 and t_3 the equality $t_1^n = t_2^n - t_3^n$ is equivalent to $t_1^n + t_3^n = t_2^n$; $t_1^n = -t_2^n + t_3^n$ is equivalent to $t_1^n + t_2^n = t_3^n$; the identity $-t_1^n = t_2^n + t_3^n$ may be satisfied only for $t_1 = t_2 = t_3 = 0$; while $-t_1^n = -t_2^n + t_3^n$ is equivalent to $t_1^n + t_3^n = t_2^n$; then $-t_1^n = t_2^n - t_3^n$ is equivalent to $t_1^n + t_2^n = t_3^n$; the equality $t_1^n = -t_2^n - t_3^n$ may be satisfied only for $t_1 = t_2 = t_3 = 0$; also $-t_1^n = -t_2^n - t_3^n$ is equivalent to $t_1^n = t_2^n + t_3^n$, where $t_1 \geq 0$, $t_2 \geq 0$ and $t_3 \geq 0$. Thus all zeros (w, x, y) of the function $g(z)$ are described by the set of all non-negative integer solutions of Equation (11) with the condition $z = \sum_{l=1}^{3} \upsilon_l(z)$ up to the symmetry transformations $(t_1, t_2, t_3) \mapsto (t_{\sigma(1)}, t_{\sigma(2)}, t_{\sigma(3)})$ and the multiplier $\nu \in \{1, 2\}$ as above, where $\sigma : \{1, 2, 3\} \to \{1, 2, 3\}$ is a bijective surjective mapping.

Next it is verified that the residue operator of the reciprocal function $f(z) := \frac{1}{g(z)}$ is non-degenerate for each zero $(w, x, y) = (t_1^n, t_2^n, t_3^n)$, $t_1, t_2, t_3 \in (\mathbf{Z}/2 \setminus \{0\})$ of $g(z)$ if such zero exists. From the proof above we know that all poles ω of the reciprocal function $f(z)$ or equivalently zeros of the function $g(z)$ are isolated points when zeros $z = \omega = (t_1^n i_{\kappa(1)} + t_2^n i_{\kappa(2)} + t_3^n i_{\kappa(3)})$ with $t_1, t_2, t_3 \in (\mathbf{Z}/2)$ exist.

The function $g(z)$ is analytic in the z-representation on the Cayley-Dickson algebra $\mathcal{A}_v$ and the reciprocal function $f(z)$ is meromorphic. In this case residue operators of meromorphic functions of Cayley-Dickson variables at point poles were defined and studied in [14, 12, 10, 11].

Calculating the (super-)derivative operator of the function $g(z)$ one obtains due to the chain rule (see also Proposition 5 above):

(12) $(dg(z)/dz).h = (d\exp[\tau_1(z)\exp(2\pi L_1 \tau_1^{1/n}(z))]/dz).\upsilon_1(h)$
$-[(d(\exp[\tau_2(z)\exp(2\pi L_2 \tau_2^{1/n}(z))])/dz).\upsilon_2(h)](\exp[\tau_3(z)\exp(2\pi L_3 \tau_3^{1/n}(z))])$
$-(\exp[\tau_2(z)\exp(2\pi L_2 \tau_2^{1/n}(z))])[(d(\exp[\tau_3(z)\exp(2\pi L_3 \tau_3^{1/n}(z))])/dz).\upsilon_3(h)]+$
$(d|z - \sum_{l=1}^{3} \upsilon_l(z)|^2 L_4/dz).(h - \sum_{l=1}^{3} \upsilon_l(h))$,

for each $z, h \in \mathcal{A}_v$, where

$$(13) \quad (d\exp[\tau_l(z)\exp(2\pi L_l \tau_l^{1/n}(z))]/dz).\upsilon_1(h) =$$
$$[\sum_{k=1}^{\infty}\sum_{m=0}^{k-1}\{\xi^m I \xi^{k-m-1}\}/k!].(I e^{2\pi L_l \vartheta}$$

$+\vartheta^n[\sum_{k=1}^{\infty}\sum_{m=0}^{k-1}\{(2\pi L_l\vartheta)^m(2\pi L_l I)(2\pi L_l\vartheta)^{k-1-m}\}/k!]].[(d\vartheta/dz).\upsilon_l(h)]$,

(14) $(d\vartheta/dz).h = [de^{(Ln\tau_l(z))/n}/dz].\upsilon_l(h) = \tau_l^{-1+1/n}(z)\tau_l(h)/n$,

(15) $(d|z-\sum_{l=1}^3\upsilon_l(z)|^m L_4/dz).(h-\sum_{l=1}^3\upsilon_l(h)) = \frac{m}{2}\alpha^{(m-2)/2}\{(h-\sum_{l=1}^3\upsilon_l(h))\{2\pi_0(z)-z+\sum_{l=1}^3\upsilon_l(z)\}+(z-\sum_{l=1}^3\upsilon_l(z))\{2\pi_0(h)-h+\sum_{l=1}^3\upsilon_l(h)\}\}L_4$ for each $m\in\mathbf{R}$, since υ_l is the real-linear projection operator and $\tau_l(z) = \tau_l(\upsilon_l(z))$ and $\pi_0(\upsilon_l(z))=0$ for each $l=1,2,3$ and $z\in\mathcal{A}_v$, where each curled bracket corresponds to the right order of multiplication $\{ab...cd\} = a(b(...(cd)...))$, the letter I denotes the unit operator acting here on ${}_lh := \upsilon_l(h)$, $\xi = \vartheta^n e^{2\pi L_l\vartheta}\in\mathbf{H}_l$, $\vartheta = \tau_l^{1/n}(z)\in\mathbf{C}_l$, $|z|^m = \exp[\frac{m}{2}Ln(z(2\pi_0(z)-z))]$, $\alpha = |z-\sum_{l=1}^3\upsilon_l(z)|^2$ in accordance with the chosen z-representation above with the help of Formulas $(2,3)$ and $2(1-3)$.

For definiteness we use the left algorithm for calculation of line integrals of functions over Cayley-Dickson algebras. Then a residue of a meromorphic function q on a domain U with a singularity at an isolated point $\xi\in\mathcal{A}_r$ is defined as

$$(16)\quad Res_{\gamma_K}(\xi,q).M := (2\pi)^{-1}\lim_{\delta\downarrow 0}(\int_{\gamma_{\delta,K}} q(z)dz)$$

whenever this limit exists, where

(17) $\gamma_{\delta,K}(t) = (\xi+\delta\chi(t)K)\subset U\setminus\{\xi\}\subset\mathcal{A}_r$,

χ is a rectifiable closed path (i.e. loop) winding one time around zero,

(18) $2\pi M = \int_\chi dLnz$,

K is a marked Cayley-Dickson number for γ such that $|K|=1$, $0<\delta$, $\gamma_K := \gamma_{1,K}$. Making the change of the variable $z\mapsto zK$ one can relate the case of $K=1$ with the case of $K\in\mathcal{A}_r\setminus\mathbf{R}$ in Formulas $(16,17)$.

Particularly it may be the circle $\chi(t) = \rho\exp(2\pi tM)$, where $\rho>0$, $M\in\mathcal{S}_r$, $t\in[0,1]$, $0<\delta\le 1$, there is not any other singular point in the closed ball $B(\mathcal{A}_r,\xi,\rho)$ of radius ρ and the center at ξ in the Cayley-Dickson algebra $\mathcal{A}_r$, (see also §1 in [14]), here $r=v=9$, where $\mathcal{S}_r := \{M\in\mathcal{A}_r:\ Re(M)=0,\ |M|=1\}$ denotes the purely imaginary unit sphere.

Let a purely imaginary Cayley-Dickson number be $M\in\mathbf{C}_1\cap\mathcal{S}_r$ and $g(\xi)=0$, then from Formulas $(12-18)$ and $\partial(\xi+\delta\rho\exp(2\pi tM)K)/\partial t =$

$\delta\rho 2\pi(M\exp(2\pi tM))K$ it follows that

$$(19)\quad Res_{\gamma_{i_{\kappa(1)}}}(\xi,f).M=[Me^{-2\pi L_1w^{1/n}}\exp(-we^{2\pi L_1w^{1/n}})][1+\frac{2\pi}{n}L_1w^{1/n}]^{-1},$$

since $\alpha(\beta\mathbf{l})=(\beta\alpha)\mathbf{l}$ for each $\alpha,\beta\in\mathcal{A}_k$ and $\mathbf{l}=i_{2^k}$ for $k\geq 2$ in accordance with Formula 1(2), the quaternion skew field is associative and the octonion algebra $alg_{\mathbf{R}}\{M,L_1,i_{\kappa(1)}\}$ is alternative, particularly for complex numbers $\alpha,\beta\in\mathbf{C}_M:=\mathbf{R}\oplus\mathbf{R}M$, also $\upsilon_l\circ\upsilon_k(z)=0$ for each $k\neq l\in\{1,2,3\}$, where $w=\tau_1(\xi)=t_1^n\neq 0$, $x=\tau_2(\xi)=t_2^n$ and $y=\tau_3(\xi)=t_3^n\in\mathbf{Z}/2$, the loop $\gamma_{i_{\kappa(1)}}$ is given by Formula (17). Symmetrically by substitution of variables for $M\in\mathbf{C}_l\cap\mathcal{S}_r$ and non-zero $w_l\neq 0$ and $g(\xi)=0$ we get

$$(20)\quad Res_{\gamma_{i_{\kappa(l)}}}(\xi,f).M=$$

$$-[Me^{-2\pi L_lw_l^{1/n}}\exp(-w_le^{2\pi L_lw_l^{1/n}})\exp(-w_ke^{2\pi L_kw_k^{1/n}})][1+\frac{2\pi}{n}L_lw_l^{1/n}]^{-1},$$

where $w_2=x$ and $w_3=y$, $(l,k)=(2,3)$ or $(l,k)=(3,2)$, for γ given by Formula (17), $w=t_l^n$, $t_l\in\mathbf{Z}/2\setminus\{0\}$. Thus $z=wi_{\kappa(1)}+xi_{\kappa(2)}+yi_{\kappa(3)}\in\mathcal{A}_v$ with $wxy\neq 0$ is a zero of the function $g(z)$ if and only if the residue operator $Res_{\gamma i_{\kappa(l)}}(\xi,f).M$ is non-degenerate by $M\in\mathbf{C}_l\cap\mathcal{S}_r$.

If $M\in\mathbf{C}_1\cap\mathcal{S}_r$ and n is even, one gets from Formula (19):

$$(21)\quad Res_{\gamma_{i_{\kappa(1)}}}(\xi,f).M+Res_{\gamma_{i_{\kappa(1)}}}(-\xi,f).M=$$

$$[2Me^{-2\pi L_1w^{1/n}}\exp(-we^{2\pi L_1w^{1/n}})]/[(4\pi^2w^{2/n}/n^2)+1],$$

if n is odd:

$$(22)\quad Res_{\gamma_{i_{\kappa(1)}}}(\xi,f).M+Res_{\gamma_{i_{\kappa(1)}}}(-\xi,f).M=M[\exp(we^{2\pi L_1w^{1/n}})+\exp(-we^{2\pi L_1w^{1/n}})$$

$$+\frac{2\pi}{n}L_1w^{1/n}(\exp(we^{2\pi L_1w^{1/n}})-\exp(-we^{2\pi L_1w^{1/n}}))]e^{2\pi L_1w^{1/n}}/[(4\pi^2w^{2/n}/n^2)+1].$$

From Formulas (21) and (22) we have, that their sum by $1/2\leq t\leq k$ for each $1/2\leq k\in\mathbf{N}/2$ is non-zero, since $e^{2\pi L_1w^{1/n}}\in\{-1,1\}$, $t\in\mathbf{N}/2$, $w=t^n$. Analogously we get this conclusion for $Res_{\gamma_{i_{\kappa(l)}}}$ with $l=2,3$ also with $[-\exp(w_le^{2\pi L_lw_l^{1/n}})\exp(w_ke^{2\pi L_kw_k^{1/n}})]$ instead of $\exp(we^{2\pi L_1w^{1/n}})$.

Now we consider an analytic change of the variable $z\in\mathcal{A}_v$ of special type:

(23) $\eta = \eta(z) := (z - \sum_{l=1}^3 \upsilon_l(z)) + \sum_{l=1}^3 \tau_l(z)^{1/n} i_{\kappa(l)}$,

taking the branch of the n-th root function $z^{1/n}$ such that $b^{1/n} > 0$ for each $b > 0$, where $\eta(z)$ is written in the z-representation due to Formulas $(2,3)$ and $2(1-3)$. Therefore, one gets the equality:

(23.1) $|\eta(z)|^2 = |z - \sum_{l=1}^3 \upsilon_l(z)|^2 + \sum_{l=1}^3 |\tau_l(z)|^{2/n}$.

The inverse transform of the Cayley-Dickson variable is:

(23.2) $z = z(\eta) := (\eta - \sum_{l=1}^3 \upsilon_l(\eta)) + \sum_{l=1}^3 \tau_l(\eta)^n i_{\kappa(l)}$.

Therefore, the function $g(z)$ can be presented as the composition of two (super-) differentiable over the Cayley-Dickson algebra $\mathcal{A}_v$ functions $g(z) = q(\eta(z))$, where

(24) $q(\eta) := \exp[(\tau_1(\eta))^n \exp(2\pi L_1 \tau_1(\eta))]$
$-(\exp[(\tau_2(\eta))^n \exp(2\pi L_2 \tau_2(\eta))])(\exp[(\tau_3(\eta))^n \exp(2\pi L_3 \tau_3(\eta))]) + |\eta - \sum_{l=1}^3 \upsilon_l(\eta)|^2 L_4$,

$q(\eta)$ is written in the η representation with the help of Formulas $(2,3)$ and $2(1-3)$. The only zeros of $q(\eta)$ are $\eta \in \mathcal{A}_v$ satisfying two conditions:

(i) $\eta = \sum_{l=1}^3 \upsilon_l(\eta)$ and

(ii) $\tau_1^n(\eta) = \tau_2^n(\eta) + \tau_3^n(\eta)$ with $\tau_l(\eta) \in T_l^n \times (\mathbf{Z}/2)$, where T_l^n denotes the set of n roots $\sqrt[n]{1}$ of the unit in the complex field $\mathbf{C}_l$.

If α and β are two Cayley-Dickson numbers so that

(iii) $\alpha = \alpha_0 + \alpha_1 M$ and $\beta = \beta_0 + \beta_1 M$ with real coordinates $\alpha_0, \alpha_1, \beta_0, \beta_1 \in \mathbf{R}$ and a purely imaginary Cayley-Dickson number $M \in \mathcal{S}_v$, $|\alpha| > 0$, then

(iv) $Ln(\alpha + \beta) = Ln(\alpha) + Ln(1 + \beta/\alpha)$,

since α and β commute in such case.

Let $x = x_0 + x_1 M$, $y = y_1 K$ and $z = z_0 + z_1 M \in \mathcal{A}_r$ be any Cayley-Dickson numbers such that x_0, x_1, y_1, z_0 and $z_1 \in \mathbf{R}$ are real; M and $K \in \mathcal{S}_r$ are purely imaginary and orthogonal $Re(MK^*) = 0$. Suppose that closed rectifiable curves $x(t)$, $y(t)$ and $z(t)$ are given, $t \in [0, 2\pi]$, such that

(v) $|z(t)| > |x(t) + y(t)|$ for each $t \in [0, 2\pi]$, where

(vi) $z = |z| e^{Mt}$ with $t \in [0, 2\pi]$, i.e. $\cos(t) = z_0/|z|$, $|z|^2 = z_0^2 + z_1^2$. Then we infer that

(vii) $x + y + z = |x + y + z| e^{L\phi}$,

where $|z|^2 = [(x_0 + z_0)^2 + (x_1 + z_1)^2 + y_1^2]^{1/2}$, $\cos(\phi) = (x_0 + z_0)/|x + y + z|$, $L = L(t) = [(x_1 + z_1)M + y_1 K]/(|x + y + z| \sin(\phi)) = \pm((x_1 + z_1)M + y_1 K)/\sqrt{(x_1 + z_1)^2 + y_1^2}$.

At first we consider the particular case when

$(vii.1)$ $y_1(t) = y_1$ is constant and K and M are constant and

$(vii.2)$ the algebra $alg_{\mathbf{R}}(M,K)$ over the real field generated by M and K is contained in an alternative sub-algebra in $\mathcal{A}_r$.

Let $\eta(t) = \alpha_0(t) + \alpha_1(t)M + y_1K$ be a curve in $\mathcal{A}_r$ with real functions $\alpha_0(t+\pi) = -\alpha_0(t)$ and $\alpha_1(t+\pi) = -\alpha_1(t)$ for each $t \in [0,2\pi]$ so that $\alpha_0^2(t) + \alpha_1^2(t) > y_1^2$ for each t, for example, when $\alpha_0(t) + \alpha_1(t)M$ is a circle with the center at zero of radius $\rho > |y_1|$. Therefore, one gets $\eta(t+\pi) = -\alpha_0(t) - \alpha_1(t)M + y_1K = -(\alpha_0(t) + \alpha_1(t)M - y_1K)$. Then we infer the equalities:

$(viii.1)$ $\int_0^{2\pi} dLn\ \eta(t) = \frac{1}{2}\int_0^{4\pi} dLn\ \eta(t) = \frac{1}{2}[\int_0^{2\pi} dLn\ \eta(t) + \int_0^{2\pi} dLn\ \eta(t+\pi)]$
$= \frac{1}{2}[\int_0^{2\pi} d\ln(\alpha_0^2+\alpha_1^2+y_1^2) + \int_0^{2\pi} d\{[\alpha_1M + y_1K]\phi/\sqrt{\alpha_1^2+y_1^2}\}$
$+ \int_0^{2\pi} d\{[\alpha_1M - y_1K](\phi+\pi)/\sqrt{\alpha_1^2+y_1^2}\}]$
$= \frac{1}{2}\int_0^{2\pi} d\ln(\alpha_0^2+\alpha_1^2+y_1^2) + \int_0^{2\pi} d\{\alpha_1M\phi/\sqrt{\alpha_1^2+y_1^2}\}$
using (vii), consequently,

$(viii.2)$ $\int_0^{2\pi} dLn\ \eta(t) \in \mathbf{R} \oplus M\mathbf{R}$.

On the other hand, we have the equality $(\alpha_0 + \alpha_1M + y_1K) = (\alpha_0K^* + \alpha_1MK^* + y_1)K$, since M and K are contained in the alternative sub-algebra in the Cayley-Dickson algebra $\mathcal{A}_r$ by our supposition made above. Then we have the equality $Ln\ (\alpha_0(t)K^* + \alpha_1(t)MK^* + y_1) = Ln\ (\alpha_0(t)K^* + \alpha_1(t)MK^*) + Ln\ (1 + y_1/(\alpha_0(t)K^* + \alpha_1(t)MK^*))$ for each $t \in [0,2\pi]$, consequently, the winding number around zero of the curve $(\alpha_0(t) + \alpha_1(t)M + y_1K)$ is equal to that of $(\alpha_0(t) + \alpha_1(t)M)$, since $|y_1|/|\alpha_0(t)K^* + \alpha_1(t)MK^*| = |y_1|/|\alpha_0(t) + \alpha_1(t)M| < 1$ for each $t \in [0,2\pi]$.

Now we consider the general case

$(vii.3)$ $M = M(t) \in \mathcal{S}_r$ and $K = K(t) \in \mathcal{S}_r$ and they are orthogonal $K \perp M$, i.e. $Re(KM^*) = 0$, for each $t \in [0,2\pi]$.

Making transformations and using Formulas (vi, vii) we deduce that

(ix) $Ln\ (x(t) + y + z(t)) - Ln\ z(t) = Ln\frac{|x(t)+y+z(t)|}{|z(t)|} + L(t)\phi(t) - Mt =$
$\frac{1}{2}Ln[1+\frac{(z_0+x_0)^2+(x_1+z_1)^2+y_1^2-z_0^2-z_1^2}{z_0^2+z_1^2}]+M[\frac{(x_1+z_1)}{\sqrt{(x_1+z_1)^2+y_1^2}}\arccos\frac{x_0+z_0}{|x+y+z|}-\arccos\frac{z_0}{|z|}]+$
$K\frac{y_1}{\sqrt{(x_1+z_1)^2+y_1^2}}\arccos\frac{x_0+z_0}{|x+y+z|}$,
where $x_0 = x_0(t)$, $x_1 = x_1(t)$, $z_0 = z_0(t)$, $z_1 = z_1(t)$.

In view of Rouché's theorem [9, 10, 11] in the complex plane $\mathbf{R} \oplus \mathbf{R}M$

(x) $\int_0^{2\pi} d[Ln\ (q(t){+}z(t)){-}Ln\ z(t)) = \int_0^{2\pi} d\{\frac{1}{2}Ln[1{+}\frac{(z_0+q_0)^2+(q_1+z_1)^2-z_0^2-z_1^2}{z_0^2+z_1^2}]+$
$M[\arccos\frac{q_0+z_0}{|q+z|} - \arccos\frac{z_0}{|z|}]\} = 0,$
where $q = q(t) = q_0(t) + q_1(t)M$ is a rectifiable closed curve (i.e. a loop: $q_0(0) = q_0(2\pi)$) such that $|q(t)| < |z(t)|$ for each $t \in [0, 2\pi]$.

Conditions $(vii.1)$ and $(vii.2)$ can be abandoned with the help of the homotopy theorem (see Theorem 2.15 in [10, 11]). It can be lightly seen that $(viii.1, x)$ imply the equality

(xi) $\int_0^{2\pi} d\{\frac{1}{2}Ln[1 + \frac{(z_0+x_0)^2+(x_1+z_1)^2+y_1^2-z_0^2-z_1^2}{z_0^2+z_1^2}]$
$+M[\frac{(x_1+z_1)}{\sqrt{(x_1+z_1)^2+y_1^2}} \arccos\frac{x_0+z_0}{|x+y+z|} - \arccos\frac{z_0}{|z|}]$
$+K\frac{y_1}{\sqrt{(x_1+z_1)^2+y_1^2}} \arccos\frac{x_0+z_0}{|x+y+z|}\} = 0,$
since $|\frac{(z_0+x_0)^2+(x_1+z_1)^2+y_1^2-z_0^2-z_1^2}{z_0^2+z_1^2}| < 1$ for each t and
$\int_0^{2\pi} dLn[1 + \frac{(z_0+x_0)^2+(x_1+z_1)^2+y_1^2-z_0^2-z_1^2}{z_0^2+z_1^2}] = 0.$

Indeed, consider the equation:

$\xi\cos(\alpha) - \sqrt{1-\xi^2}\sin(\alpha) = \eta,$

where $\alpha \in \mathbf{R}$, $\eta \in \mathbf{R}$, $|\eta| \le 1$. This gives $(\xi\cos(\alpha){-}\eta)^2 = (1{-}\xi^2)\sin^2(\alpha)$ or $\xi^2{-}2\xi\eta\cos(\alpha){+}(\eta^2{-}\sin^2(\alpha)) = 0$. By Vieta's formula one gets two solutions $\xi_{1,2} = \eta\cos(\alpha) \pm \sqrt{1-\eta^2}\sin(\alpha)$. Particularly we take $\eta = \frac{1}{2\pi}\arccos(\frac{z_0}{|z|})$, $\cos(\alpha) = \frac{x_1+z_1}{\sqrt{(x_1+z_1)^2+y_1^2}}$, $\sin(\alpha) = \frac{y_1}{\sqrt{(x_1+z_1)^2+y_1^2}}$. Therefore, we deduce that

$M[\frac{x_1+z_1}{\sqrt{(x_1+z_1)^2+y_1^2}} \arccos(\frac{x_0+z_0}{|x+y+z|}){-}\arccos(\frac{z_0}{|z|})]{+}K\frac{y_1}{\sqrt{(x_1+z_1)^2+y_1^2}} \arccos(\frac{x_0+z_0}{|x+y+z|})$
$= M[\arccos(\frac{x_0+z_0}{|x+y+z|}){-}\arccos(\frac{z_0}{|z|})]\cos(\alpha){+}[K\arccos(\frac{x_0+z_0}{|x+y+z|}){-}2\pi M\sqrt{1-\eta^2}]\sin(\alpha).$

The transformation $\begin{pmatrix}\xi\\ \sqrt{1-\xi^2}\end{pmatrix} \mapsto \begin{pmatrix}\cos(\alpha) & -\sin(\alpha)\\ \sin(\alpha) & \cos(\alpha)\end{pmatrix}\begin{pmatrix}\xi\\ \sqrt{1-\xi^2}\end{pmatrix}$ is orthogonal. In Formula (x) one can choose $\frac{x_0+z_0}{|x+y+z|} = \frac{q_0+z_0}{|q+z|}$ with $|q| = |x+y|$ and $Im(q) \in Im(z)\mathbf{R}$, where $Im(q) = q - Re(q)$, $|x+y|^2 = |x|^2 + |y_1|^2$, since x and y are orthogonal, $x \perp y$. If $y_1 = 0$, then $|\cos(\alpha)| = 1$ and $\sin(\alpha) = 0$. If $|y_1| > 0$, then $|\cos(\alpha)| < 1$ and $|\sin(\alpha)| > 0$. Using the homotopy theorem and Formulas $(viii.1, viii.2, x)$ we infer that

$(xi.1)$ $\int_0^{2\pi} d\{M[\arccos(\frac{x_0+z_0}{|x+y+z|}) - \arccos(\frac{z_0}{|z|})]\cos(\alpha)$
$+[K\arccos(\frac{x_0+z_0}{|x+y+z|}) - M\sqrt{(2\pi)^2 - \arccos^2(\frac{z_0}{|z|})}]\sin(\alpha)\}(t) = 0,$
since the function in the integral does not change its branch, consequently, the function $[Ln\ (x(t) + y(t) + z(t)) - Ln\ z(t)]$ does not change its branch.

Thus Formula $(xi.1)$ implies that

(xii) $\hat{I}n(0, x+y+z) = \hat{I}n(0, z) := \frac{1}{2\pi}\int_0^{2\pi} dLn\ z(t),$

when Conditions $(v, vi, vii.3)$ are satisfied (see also Corollary 3.26.3 in [10, 11]), where $\frac{1}{2\pi}\int_0^{2\pi} dLn\ z(t) = P \in \mathbf{Z}\mathcal{S}_r$, particularly $P = M$ for a constant $M \in \mathcal{S}_r$ (see above).

If $u = (u_1, ..., u_k)$ is a vector in the Euclidean space $\mathbf{R}^k$, then $|u|_p := \sqrt[p]{u_1^p + ... + u_k^p}$ for $1 \le p < \infty$. Let $t > s > 0$ and $0 < a \le b$, then

(25) $(a^t + b^t)^{1/t} < (a^s + b^s)^{1/s}$.

Indeed, Inequality (25) is equivalent to $(1+x^t)^{1/t} < (1+x^s)^{1/s}$ for each $x \ge 1$ or to

(25.1) $(1 + y^p)^{1/p} < 1 + y$,

where $x = b/a,\ \ p = t/s,\ \ y = x^s \ge 1$. Consider the function $g(y) = (1 + y^p)^{1/p} - (1+y)$. One has $g(1) = 2^{1/p} - 2 < 0$ and $g'(y) = (1+y^p)^{-1+1/p}y^{p-1} - 1 < 0$ for each $y \ge 1$, since $y^p < 1 + y^p \quad \Leftrightarrow \quad y^{p-1} < (1 + y^p)^{(p-1)/p} \Leftrightarrow (1+y^p)^{-1+1/p}y^{p-1} - 1 < 0$. Therefore, $g(y) < 0$ for each $y \ge 1$, which implies Inequality (25).

Particularly,

(26) $c^2 < a^2 + b^2$ and

(27) $c^{2n} > a^{2n} + b^{2n}$ for any non-zero real numbers $a, b, c \ne 0$ such that $|c|^n = |a|^n + |b|^n$ with $n \ge 3$.

Since $2^n > 1^n + 1^n$ and $3^n > 2^{n+1}$ for $n \ge 3$, we consider the equation $a^n + b^n = c^n$ for half-integer numbers $a, b, c \in (\mathbf{Z}/2)\setminus\{0\}$ with $\max(|a|, |b|, |c|) \ge 3$. That is in the ball $B(\mathcal{A}_v, 0, 2^{n+1})$ in the Cayley-Dickson algebra Equation (10) with non-zero half-integer numbers t_1, t_2, t_3 has not any solution, i.e. the function $g(z)$ is non-zero, $g(z) \ne 0$, for the argument $z = t_1^n i_{\kappa(1)} + t_2^n i_{\kappa(2)} + t_3^n i_{\kappa(3)} \in B(\mathcal{A}_v, 0, 2^{n+1})$ with $t_1t_2t_3 \ne 0$, since $|z|^2 = t_1^{2n} + t_2^{2n} + t_3^{2n}$ for such z, where $B(\mathcal{A}_v, \xi, \rho) := \{z \in \mathcal{A}_v :\ |z - \xi| \le \rho\},\ \ \xi \in \mathcal{A}_v,\ \ \rho > 0$.

We proceed by induction. Suppose that in the ball $B(\mathcal{A}_v, 0, 2\rho^n)$ the function $g(z)$ is no-zero for $z = c^n i_{\kappa(1)} + a^n i_{\kappa(2)} + b^n i_{\kappa(3)} \in B(\mathcal{A}_v, 0, 2\rho^n)$ with $abc \ne 0,\ \ a, b, c \in (\mathbf{Z}/2) \setminus \{0\}$, where $\rho \in \mathbf{N} := \{1, 2, 3, ...\}$ is a natural number.

We take the path $\kappa(t)$ consisting of two circles γ^1 and γ^2 of radius $(\rho+1+\epsilon)^n$ and $(\rho+\epsilon)^n$, where $0 < \epsilon < 1/4$, and a joining them path ω gone twice in one and the opposite direction such that $\kappa(t)$ contains no any Cayley-Dickson number z with half-integer coordinates z_j for any $j = 0, ..., 2^v - 1$. That is,

$\kappa(t) = \gamma^1(4t)$ for $0 \le t < 1/4$, $\kappa(t) = \omega(4t-1)$ for $1/4 \le t < 1/2$, $\omega(0) = \gamma^1(1)$, $\kappa(t) = \gamma^2(3-4t)$ for $1/2 \le t < 3/4$, $\kappa(t) = \omega(4-4t)$ for $3/4 \le t \le 1$, $\omega(1) = \gamma^2(0)$, where $\gamma^1(t) = (\rho+1+\epsilon)^n \chi(t) i_{\kappa(1)}$, $\gamma^2(t) = (\rho+\epsilon)^n \chi(t) i_{\kappa(1)}$, $\chi(t) = \exp(2\pi t M)$, where $\rho > 0$, $M = i_1$, $t \in [0,1]$.

In view of the theorem about change of a variable in the line integral over the Cayley-Dickson algebra $\mathcal{A}_v$ and the theorem about residues (see Theorems 2.6, 2.11 and 2.13 in [14] and [10, 11]) we infer that

$$(28) \quad \int_\kappa f(z)dz = \int_\psi \frac{1}{q(\eta)} d\eta,$$

where $\psi(t) = \eta(\kappa(t))$ for each $t \in [0,1]$, consequently,

$$(29) \quad \sum_{\xi = w_1^n i_{\kappa(1)} + w_2^n i_{\kappa(2)} + w_3^n i_{\kappa(3)};\ (\rho+\epsilon) \le |c| < (\rho+1+\epsilon);\ c^n = a^n + b^n} Res_{\gamma_{i_{\kappa(1)}}}(\xi, f).M$$

$$= \sum_{\xi = w_1^n i_{\kappa(1)} + w_2^n i_{\kappa(2)} + w_3^n i_{\kappa(3)};\ (\rho+1+\epsilon)^2 > w_l^2 + w_k^2;\ |c| \ge (\rho+\epsilon);\ c^n = a^n + b^n} Res_{\gamma_{i_{\kappa(1)}}}(\xi, f).M$$

due to Formulas $(23-24)$ and (i, ii, xii) above, where $|c| = \max(|w_1|, |w_2, |w_3|) = |w_m|$, these numbers are ordered as $|w_l| \le |w_k| < |c|$, $c = w_m$, $a = w_l$, $b = w_k$, $a, b, c \in \mathbf{Z}/2$, l, k, m are pairwise distinct, $l \ne k \ne m$, $l \ne m$; l, k and $m \in \{1,2,3\}$, since the algebra $alg_{\mathbf{R}}(M, i_{\kappa(2)}, i_{\kappa(3)})$ is alternative.

But Formulas $(26-29)$ give us the contradiction with the supposition that there are half-integer solutions $c^n = a^n + b^n$ for $n \ge 3$ with the non-zero product $abc \ne 0$, since $0 < \epsilon < 1/4$ may be arbitrary small and the limits $\lim_{\downarrow\epsilon}$ of the right and the left sides of (29) coincide. Thus in the ball $B(\mathcal{A}_v, 0, 2(\rho+1)^n)$ there is not any solution of the equation $g(\xi) = 0$ with $\xi = w_1^n i_{\kappa(1)} + w_2^n i_{\kappa(2)} + w_3^n i_{\kappa(3)}$ and $w_1, w_2, w_3 \in (\mathbf{Z}/2) \setminus \{0\}$, since $|\xi|^2 = a^{2n} + b^{2n} + c^{2n} = w_1^{2n} + w_2^{2n} + w_3^{2n}$. The proof above leads to the conclusion that Equation (11) has not any solution in natural numbers $a, b, c \in \mathbf{N}$ for $n \ge 3$, since $\rho \ge 2$ is arbitrary.

8. Remark. Alternatively it is possible to use the argument principle and the Cayley-Dickson analogs of Rouché's theorem and the homotopy theorem instead of residues in §7. For this purpose additional functions can be used.

Denote by $\theta_{k,l} : \mathbf{H}_k \to \mathbf{H}_l$ isomorphisms of copies of the quaternion skew field. Making the substitution of variables and using the isomorphisms $\theta_{k,l}$

and the equation $w^n = x^n + y^n$, we write two new functions g_1 and g_2 which have the same zeros as g:

(1) $g_1(z) = g(z - \upsilon_2(z) - \upsilon_3(z) + [(\theta_{1,2}(\tau_1(z)) - (\theta_{3,2}(\tau_3(z))]i_{\kappa(2)} + [(\theta_{1,3}(\tau_1(z)) - (\theta_{2,3}(\tau_2(z))]i_{\kappa(3)})$ and

(2) $g_2(z) = g(z - \upsilon_1(z) - \upsilon_2(z) + [(\theta_{2,1}(\tau_2(z)) + (\theta_{3,1}(\tau_3(z))]i_{\kappa(1)} + [(\theta_{1,2}(\tau_1(z)) - (\theta_{3,2}(\tau_3(z))]i_{\kappa(2)})$ and

(3) $g_3(z) = g(z - \upsilon_1(z) - \upsilon_3(z) + [(\theta_{2,1}(\tau_2(z)) + (\theta_{3,1}(\tau_3(z))]i_{\kappa(1)} + [(\theta_{1,3}(\tau_1(z)) - (\theta_{2,3}(\tau_2(z))]i_{\kappa(3)})$

and analogous transformed functions $q_l(\eta)$, $l = 1, 2, 3$.

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